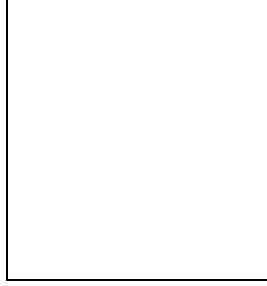# PREDICTIONS OF PHYSICAL OBSERVABLES FROM MINIMAL NEUTRINO STRUCTURES

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We find all possible seesaw textures which can describe in a natural way the currently observed neutrino oscillation pattern in terms of a minimum number of parameters. Natural here means due only to the relative smallness (vanishing) of some parameters in the relevant lagrangian, without special relations or accidental cancellations among them. The corresponding predictions for the mixing angle θ_{13} and the effective mass m_{ee} are given.

1 Introduction

In the last five years the neutrino physics has seen remarkable experimental developments. Observations of atmospheric [1], solar [2], beam [3] and reactor [4] neutrino deficits have established that at least two of the neutrinos have a mass. All these results can be nicely accommodated in the simple 3 neutrino $(\nu_e, \nu_\mu, \nu_\tau)$ framework with 2 mass differences and 3 mixing angles, determined with the values:

$$2.3 \cdot 10^{-3}\,\mathrm{eV}^2 < \delta m_{32}^2 < 3.1 \cdot 10^{-3}\,\mathrm{eV}^2\,, \quad 6.4 \cdot 10^{-5}\,\mathrm{eV}^2 < \delta m_{21}^2 < 7.8 \cdot 10^{-5}\,\mathrm{eV}^2\,,$$
$$0.7 < \tan^2\theta_{23} < 1.3\,, \quad 0.35 < \tan^2\theta_{12} < 0.55\,, \quad \sin^2\theta_{13} \lesssim 0.06\,. \tag{1}$$

A real pattern of neutrino masses and mixings begins to be therefore at hand. Moreover, in the future, 3 additional parameters could be measured if not too tiny: the mixing angle θ_{13} in neutrino superbeam and factory experiments [5], the CKM type CP-violating phase δ at neutrino factories and the m_{ee} combination of masses and mixings in the $0\nu2\beta$ decay experiments. For a hierarchical spectrum of neutrino masses, either "normal" $(m_3 \gg m_2 > m_1)$ or "inverted" $(m_1 \simeq m_2 \gg m_3)$, as we shall consider in the following[a], one can tentatively assume that all the

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[a]The degenerate spectrum is not considered since it does not satisfy the naturalness criterion as defined below.

six oscillation observables will be measured, some of them with significant precision, perhaps together with the $0\nu 2\beta$ mass m_{ee}. This makes a total of seven observables, which a theory of neutrino masses should be able to predict. In view of future experiments it would be crucial that, at the least, we could predict θ_{13}, m_{ee} or δ from the values of the other parameters.

Nevertheless, to predict such experimentally testable relations between these seven observables turns out to be a very difficult task. The seesaw mechanism provides a natural explanation for the smallness of the neutrino masses but it does not explain the above pattern of neutrino masses and mixings. A simple way to illustrate this problem is to perform a counting of the parameters in the seesaw extended standard model based on the following lagrangian with three heavy right handed Majorana neutrinos N_{iR} (after electroweak symmetry breaking):

$$\mathcal{L} \ni -\frac{1}{2} N_R^T M_R N_R - v \bar{N}_R Y_\nu N_L + h.c. \,, \tag{2}$$

where Y_ν is the neutrino Yukawa coupling matrix, with $N_R^T = (N_1, N_2, N_3)_R$, $N_L^T = (\nu_e, \nu_\mu, \nu_\tau)_L$, and $v = 174$ GeV. The associated neutrino mass matrix in the flavour basis is given by

$$M_\nu = -U_l^T Y_\nu^T M_R^{-1} Y_\nu U_l v^2 \,, \tag{3}$$

with U_l the mixing matrix resulting from the diagonalization of the charged lepton mass matrix. In the basis where the charged lepton and right handed neutrino mass matrices are real and diagonal, in addition to the three right handed neutrino masses, there are 9 real parameters and 6 phases in the Yukawa coupling matrix Y_ν, which give a total of 18 unknown parameters. These 18 parameters must necessarily be known in order to really test the seesaw mechanism and the underlying flavour structure, and there are only seven observables!

A simple minded conjecture which allows to correlate some of the 7 observables, is that these correlations could arise from the economy in the number of independent basic parameters. As the simplest example, here one can consider the possibility that the lagrangian has a maximum number of negligible small entries; we insist on the fact that the vanishing parameters have to be those of the basic lagrangian, not of the neutrino mass matrix which in the seesaw model are only combinations of the fundamental parameters. Such a conjecture may offer an opportunity to solve the above flavour problem. We believe in particular that to address this difficult problem, it is important to determine in a systematic way how complex must be the minimal flavour structures to explain the known data. More generally this also provides one of the rare chance to test indirectly the seesaw origin of the neutrino masses, because in this case the correlations obtained will be in general closely related to the seesaw structure of the neutrino mass matrix. Note that there are a priori several mechanisms, including symmetries, which could be responsible for the relative smallness (or vanishing) of some parameters. The purpose of this work is not to identify these mechanisms but to see what are the consequences of assuming that such a mechanism exists.

To find all correlations between observables assuming vanishing entries in the basic seesaw lagrangian would require examining a huge number of different possibilities. In the following we shall consider only the possibilities describing the currently observed pattern of the data (1) in a "natural" way, i.e. only by the relative smallness (vanishing) of some parameters in $\mathcal{L}$, barring special relations or accidental cancellations among them. In particular this should be the case when accounting simultaneously for the smallness of $R \equiv \delta m_{21}^2/\delta m_{32}^2$ and for the largeness of θ_{23}, the most peculiar feature of the data so far, even though an accidental cancellation might also produce the same feature.

2   The number of parameters and U_l rotation issues

Assuming vanishing entries in Eq. 2, one important preliminary remark which must be done is that there are always more parameters in the basic seesaw lagrangian than in the light neutrino

mass matrix. This comes from the fact that M_ν depends only on $\sim Y_\nu^2/M_N$ and not on the normalization of the lines of Y_ν and of the M_N separately. Formally writing Y_ν and M_N as

$$Y_\nu = DA\,, \qquad M_N = D\mu D\,, \tag{4}$$

with $D = \text{diag}(d_1, d_2, d_3)$ an adimensional diagonal matrix taken in such a way that the lines of A are normalized to one, this manifests itself by the fact that, for fixed values of A and μ, the neutrino mass matrix is independent of D. And what determines the number of correlations among physical observables is the number of independent "effective" parameters in the neutrino mass matrix, not in the lagrangian. Since there are 6 real observables, if the neutrino mass matrix has n effective real parameters, there will be $6 - n$ relations between the parameters. In practice one can convince oneself that, in order to reproduce the known data on δm_{23}^2, δm_{12}^2, θ_{23}, θ_{12} and θ_{13} along the assumptions we made, we need at least 4 effective parameters. In the following we have determined all minimal (i.e. with 4 effective parameters) configurations which are not already excluded by experiment, that is to say which give 2 predictions, one for θ_{13} and one for m_{ee} as a function of δm_{23}^2, δm_{12}^2, θ_{23} and θ_{12} (and in one case also as a function of the phase δ).

Before presenting the results, note also that if we restrict ourselves to 4 effective parameters, the charged lepton mixing matrix must have a simple structure. The only forms which are possible [6] are a simple rotation along one of the flavour e, μ or τ axis, i.e. $U_l = R_{23}, R_{12}, R_{13}$, or a double rotation $U_l = R_{12}R_{23}, R_{13}R_{23}$. Any other rotation would bring too many effective parameters to the neutrino mass matrix or, like for example $R_{23}R_{12}$, would lead to an already excluded value of θ_{13}.

3 Results

Following a methodology described in Ref. [6], we find that there are only five definite testable sets of predictions not already excluded by the data, given in Table 1. There are only four cases (A, B, C, D) that allow to connect $\sin\theta_{13}$ and m_{ee} with θ_{23}, θ_{12} and $R \equiv \delta m_{32}^2/\delta m_{21}^2$, whereas in case E the correlation also involves the CP-violating phase δ. The relations quoted in Table 1

Table 1: Summary of the possible correlations between θ_{13}, m_{ee} and θ_{23}, θ_{12}, δ, $R = \delta m_{32}^2/\delta m_{21}^2$ (with $m_{\text{atm}} \equiv \sqrt{|\delta m_{32}^2|}$). The columns "$N_R$" and "$U_l$" give the number of right handed neutrinos involved in the see-saw realization of each case and the form of the U_l rotation matrix. An inverse hierarchy is obtained only in the case E. Cases A2, B2, E2, are obtained from A1, B1, E1, with the replacements $\tan\theta_{23} \to \cot\theta_{23}$ and $\cos\delta \to -\cos\delta$.

	$\sin\theta_{13}$	$|m_{ee}|/m_{\text{atm}}$	N_R	U_l
A1	$\dfrac{1}{2}\tan\theta_{23}\sin 2\theta_{12}\sqrt{R}$	$\sin^2\theta_{12}\sqrt{R}$	2	$\mathbf{1}$
B1	$\dfrac{1}{2}\tan\theta_{23}\tan 2\theta_{12}\,(R\cos 2\theta_{12})^{1/2}$	0	3	$\mathbf{1}$
C	$\dfrac{1}{2}\tan 2\theta_{12}\,(R\cos 2\theta_{12})^{3/4}$	0	3	R_{23}
D	$\dfrac{1}{2}\dfrac{\tan 2\theta_{12}}{|\tan 2\theta_{23}|}(R\cos 2\theta_{12})^{1/2}$	$\left(\dfrac{\sin\theta_{13}}{\cos 2\theta_{23}}\right)^2$	3	$\mathbf{1}$
E1	$-\dfrac{\tan\theta_{23}}{\cos\delta}\dfrac{1-\tan\theta_{12}}{1+\tan\theta_{12}}$	$2\cot\theta_{23}\sin\theta_{13}$	2, 3	$R_{12}(R_{23})$

Table 2: Parameters for the see-saw cases with 2 right handed neutrinos of Table 1. A is the Dirac neutrino mass matrix with $(AA^\dagger)_{ii}$ normalized to unity and μ is related to the right handed neutrino mass matrix by Eq. 4. The lagrangians leading to the corresponding predictions of Table 1 are given by injecting A and μ below in Eqs. 2 and 4 with D any diagonal matrix and with μ_0 any mass scale. ϵ and σ denote small entries relative to unity. c, s or c', s' denote the cosine and the sine of arbitrary angles, θ and θ'. U_l is the rotation matrix of the left handed charged leptons. Case E1 (E2) can also be obtained from the E1 (E2) configurations below with $c = 1$, $s = 0$, $U_l = R_{12}R_{23}$ ($U_l = R_{13}R_{23}$) and with the small entry in μ_{11}, μ_{22}, A_{12} or A_{21}.

	μ/μ_0	A	U_l
A1	$\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & s & c \\ c' & s'e^{i\phi} & 0 \end{pmatrix}$	1
A2	$\begin{pmatrix} \epsilon & 0 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & s & c \\ c' & 0 & s'e^{i\phi} \end{pmatrix}$	1
E1 (E2)	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c & s \end{pmatrix}$ + 1 small entry	R_{12} (R_{13})

are obtained through an expansion in R. The higher orders in the expansion are suppressed by $R^{1/2}$ in all cases except E, where the leading corrections to the relations in Table 1 are of order R. In all cases, anyhow, the exact relations can be obtained from Tables 2 and 3, where the sets of parameters that originate these correlations are shown. Table 1 specifies the number of right handed neutrinos involved and it also gives the form of the rotation on the charged lepton sector. For the cases with two neutrinos, the third neutrino is assumed to be very massive and/or to have negligible couplings. E is the only case that leads to an "inverted" spectrum. For cases A1, B1, E1, the independent possibility exists where $\tan\theta_{23} \to \cot\theta_{23}$, $\cos\delta \to -\cos\delta$, denoted in the following by A2, B2, E2, respectively. Case A is discussed in Ref. [7,8]. The prediction for θ_{13} in case B was already obtained approximately from other phenomenological assumptions [9]. For related works see also Refs.[12] and references therein.

Given the present knowledge of θ_{23}, θ_{12} and δm_{21}^2, including the recent Kamland result [4], the ranges of values for $\sin\theta_{13}$ are shown in Fig. 1 at 90% confidence level for the different cases. It is interesting that all the ranges for $\sin\theta_{13}$, except in case D, are above $\simeq 0.02$ and some can saturate the present limit. Long-baseline experiments should explore a significant portion of this range while reducing at the same time the uncertainties of the different predictions at about 10% level[5]. Note that, in cases E, although the determination of $\sin\theta_{13}$ requires the knowledge of the CP violating phase as well, the allowed range is still limited, being $\sin\theta_{13} \gtrsim 0.10$. Furthermore, the requirement of not exceeding the present experimental bound on $\sin\theta_{13}$ gives a lower bound on $|\cos\delta|$ (and therefore an upper limit on CP-violation) that we can quantify as $|\cos\delta| > 0.8$ (at 90 % CL) given the present uncertainties. Notice that $\cos\delta < 0$ (> 0) in case E1 (E2). Verifying the prediction for $\sin\theta_{13}$ in case D would require the measurement of $\theta_{23} \neq 45°$; a bound on $|1 - \sin^2 2\theta_{23}|$ only sets an upper bound on $\sin\theta_{13}$, as shown in Fig. 1.

While the predictions for $\sin\theta_{13}$ are in an experimentally interesting range, the expectations for the $0\nu2\beta$-decay effective mass are mostly on the low side, except, as expected [10], in the only inverted hierarchical case E. The ranges for each individual cases with non-vanishing m_{ee} are shown in Fig. 2. The challenge of detecting a non zero m_{ee}, when applicable, is therefore harder than for $\sin\theta_{13}$, with a better chance for the only inverted hierarchical case E.

Note that we also found 3 configurations which predict $\theta_{13} = m_{ee} = 0$ which we don't show because they are not testable. Therefore, if for a large majority of the configurations an experimentally accessible value of $\sin\theta_{13}$ is found, the existence of these configurations does not
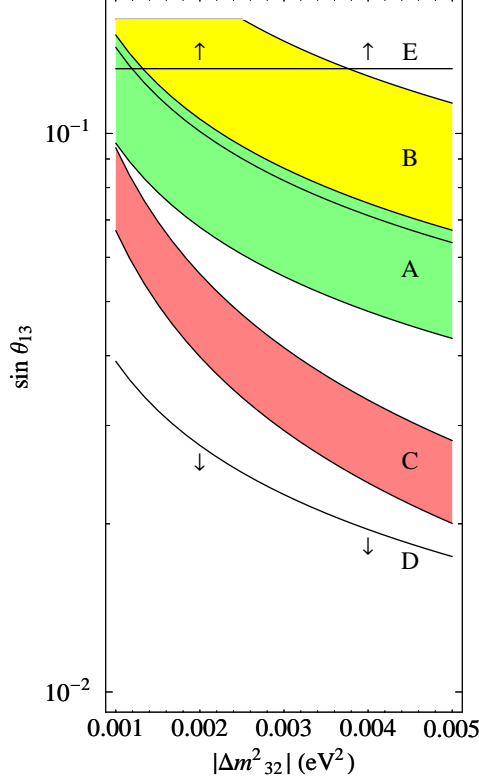
Figure 1: Ranges of values for $\sin\theta_{13}$ at 90% confidence level for the different cases, plotted as a function of δm^2_{32}. Cases D, E, which only give a bound on $\sin\theta_{13}$, are shown with a double arrow.
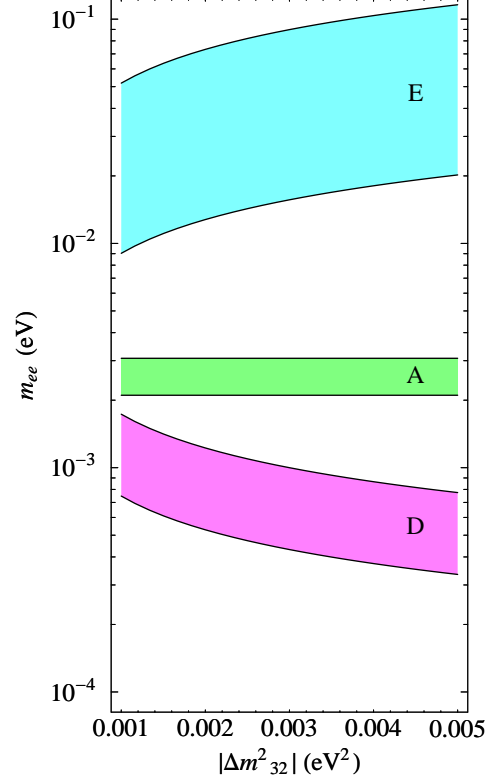
Figure 2: Ranges of values for m_{ee} at 90% confidence level. Cases B and C predict $m_{ee} = 0$.

allow us to say that such a large value is a prediction of our assumptions.

Finally note that in a model like the triplet seesaw model [11], the neutrino mass matrix is a fundamental quantity since it is directly proportional to the lepton-lepton-triplet coupling matrix. In this case, following our assumptions, we find [6] that there are only two sets of relations, those of cases C and E, which are testable and not excluded by the present data.

4 Summary

The economy in the number of basic parameters could be at the origin of some correlations between the physical observables in the neutrino mass matrix. At the present state of knowledge, the variety of the possibilities for the basic parameters themselves is large. Finding the minimal cases that describe the present pattern of the data in a natural way could be a first step in the direction of discriminating the relevant $\mathcal{L}$. This we have done with the results summarized in Tables 1–3 and illustrated in Figs. 1 and 2. It is remarkable that the number of possible correlations between the physical observables is limited (Table 1), with a relatively larger number of possibilities for the basic parameters (Tables 2–3). The relatively best chance for selecting experimentally one out of the few relevant cases is offered by $\sin\theta_{13}$. Combining this with independent studies of leptogenesis or of lepton flavour violating effects could lead to the emergence of an overall coherent picture. We have insisted on "naturalness" both in solving the "large θ_{23}-small R" problem for the normal hierarchy case and in obtaining a significant deviation of θ_{12} from $45°$, with small R, in the inverted hierarchy case. Explaining these features in a natural way offers a possible interesting guidance for model building.

Table 3: Parameters for the see-saw cases with 3 right handed neutrinos of Table 1. ϵ and σ denote small positive quantities, while $a = \mathcal{O}(1)$. μ, A and U_l as in Table 2. Case E1 (E2) can also be obtained from the E1 (E2) configurations below with $c = 1$, $s = 0$ and $U_l = R_{12}R_{23}$ ($U_l = R_{13}R_{23}$).

	μ/μ_0	A	U_l
B1 (B2)	$\begin{pmatrix} a & c & s \\ c & 0\,(\epsilon e^{i\phi}) & 0 \\ s & 0 & \epsilon e^{i\phi}\,(0) \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\mathbf{1}$
	$\begin{pmatrix} 0 & c & s \\ c & 0\,(\sigma) & 0 \\ s & 0 & \sigma\,(0) \end{pmatrix}$	$\begin{pmatrix} c' & s'e^{i\phi} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} c' & 0 & s'e^{i\phi} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\mathbf{1}$
	$\begin{pmatrix} ae^{i\phi} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1\,(0) & 0\,(1) \\ 0 & s & c \end{pmatrix}$	$\mathbf{1}$
	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix}$	$\begin{pmatrix} c' & s'e^{i\phi} & 0 \\ 0 & 1\,(0) & 0\,(1) \\ 0 & s & c \end{pmatrix}, \begin{pmatrix} c' & 0 & s'e^{i\phi} \\ 0 & 1\,(0) & 0\,(1) \\ 0 & s & c \end{pmatrix}$	$\mathbf{1}$
C	$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & \sigma \end{pmatrix}$	$\begin{pmatrix} c' & 0 & s'e^{i\phi} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	R_{23}
D	$\begin{pmatrix} a & c & s \\ c & 0 & \epsilon e^{i\phi} \\ s & \epsilon e^{i\phi} & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\mathbf{1}$
	$\begin{pmatrix} 0 & c & s \\ c & 0 & \sigma \\ s & \sigma & 0 \end{pmatrix}$	$\begin{pmatrix} c' & s'e^{i\phi} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} c' & 0 & s'e^{i\phi} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\mathbf{1}$
E1 (E2)	$\begin{pmatrix} 0 & \sigma & 0 \\ \sigma & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 0 & c & s \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & c & s \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & c & s \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$R_{12}\,(R_{13})$

Acknowledgments

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